

THE HOMOTOPY SEQUENCE OF NORI'S FUNDAMENTAL GROUP

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ABSTRACT. In this paper, we investigate the necessary sufficient conditions for the exactness of the homotopy sequence of Nori's fundamental group and apply these to various special situations to regain some classical theorems and give a counter example to show the conditions are not always satisfied. This work partially bases on the earlier work of H.Esnault, P.H.Hai , E.Viehweg.

1. INTRODUCTION

If $f : X \rightarrow S$ is a separable proper morphism with geometrically connected fibres between locally noetherian connected schemes, $x \rightarrow X$ is a geometric point with image $s \rightarrow S$, Grothendieck shows in [Gr, Exposé X, Corollaire 1.4] that one has a homotopy exact sequence for the étale fundamental group:

$$\pi_1^{\text{ét}}(\bar{X}_s, x) \rightarrow \pi_1^{\text{ét}}(X, x) \rightarrow \pi_1^{\text{ét}}(S, s) \rightarrow 1.$$

A similar case is that one can take X, Y to be two locally noetherian connected k -schemes with $k = \bar{k}$ and suppose Y is proper over k , so if K is an algebraically closed field containing k and if we take a K -point $z = (x, y) : \text{Spec}(K) \rightarrow X \times_k Y$, then we get a canonical morphism of topological groups

$$\pi_1^{\text{ét}}(X \times_k Y, z) \rightarrow \pi_1^{\text{ét}}(X, x) \times \pi_1^{\text{ét}}(Y, y).$$

Again Grothendieck shows in [Gr, Exposé X, Corollaire 1.7] that the canonical homomorphism is an isomorphism. This is called the Künneth formula for the étale fundamental group. If we take K to be k itself then the Künneth formula is a direct consequence of the homotopy exact sequence. If $k \subsetneq K$ then one has to apply the following base change theorem [Gr, Exposé X, Corollaire 1.8] to reduce to the case when $k = K$. The base change theorem states that the canonical map

$$\pi_1^{\text{ét}}(X \times_k K, x \times_k K) \rightarrow \pi_1^{\text{ét}}(X, x)$$

is an isomorphism between topological groups. Note that the base change theorem can be thought of as a special case of the Künneth formula by taking $Y = \text{Spec}(K)$, but it is not a corollary for logical reasons.

Let X be a reduced connected scheme over a field k , $x \in X(k)$ be a rational point. If we set $N(X, x)$ to be the category whose objects consist of triples (P, G, p) (where P is an

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FPQC G -torsor over X , G is a finite group scheme, $p \in P(k)$ is a k -rational point lying over x), whose morphisms are morphisms of X -schemes which are compatible with the group action. M.Nori proved in [Nori, Part I, Chapter II, Proposition 2] that the projective limit $\varprojlim_{N(X,x)} G$ exists in the category of k -group schemes (in the projective system we associate to each index (P, G, p) the group G). Then he defined the fundamental group $\pi^N(X, x)$ to be the projective limit $\varprojlim_{N(X,x)} G$ which is called Nori's fundamental group nowadays. If X is in addition proper over k and if k is perfect, Nori gave in [Nori, Part I, Chapter I] a Tannakian description of his fundamental group: he defined $\pi^N(X, x)$ to be the Tannakian group of the neutral Tannakian category of $\text{Ess}(X)$ (the essentially finite vector bundles on X) with the fibre functor $x^* : V \mapsto V|_x$, and he showed that this definition is the same as the one defined by the projective limit. If X is smooth instead of proper and k is a perfect field of characteristic $p > 0$, H.Esnault and A.Hogadi gave another Tannakian description of Nori's fundamental group in [EH][Section 3 and 4]. They defined $\pi^N(X, x)$ to be the Tannakian category of finite generalized stratified bundles with the fibre functor $x^* : (V_i, \sigma_i, i \geq 0) \mapsto V_0|_x$, and they showed that this definition coincide with the one defined via projective limit.

The main purpose of this article is to study the analogues of the homotopy sequence and Künneth formula for Nori's fundamental group. Since in Nori's fundamental group we only deal with rational points, if the homotopy sequence is exact then Künneth formula always automatically follows. Another special thing for Nori's fundamental group is that we always assume our schemes have rational points, so base change can not be thought of as a special case of the Künneth formula, it should be proved or disproved independently.

In [EHV][Section 2] H.Esnault, P.H.Hai, E.Viehweg, give a counter example which shows that homotopy sequence of Nori's fundamental group is not always exact even for $X \rightarrow S$ projective smooth and S projective smooth as well. And then they give a necessary and sufficient condition for the exactness of the homotopy sequence of Nori's fundamental group under the assumption that S is a proper k -scheme. But unfortunately there is a gap in the argument. In this article, our first goal is to reformulate some similar conditions to make everything work. These works are contained in Theorem 2.2 and Theorem 3.1, where we correct the mistake, improve the arguments and make the wonderful ideas hidden in that article right and clean. The upshot is that in Theorem 2.2 we don't have to assume S to be proper, so the result applies to the general definition of Nori's fundamental group.

Then we make two applications of Theorem 2.2 and Theorem 3.1. We first apply the criterion to show that the homotopy sequence for the étale quotient of Nori's fundamental group is exact. The argument is independent of Grothendieck's theory of the étale fundamental group which was developed in [Gr, Exposé X], so it can be seen as new proof of the homotopy exact sequence for the étale fundamental group (in the language of Nori's fundamental group).

In [MS][Theorem 2.3] V.B.Mehta and S.Subramanian proved that Künneth formula holds for Nori's fundamental group if both X and Y are proper k -schemes. In §3, we apply Theorem 3.1 to give a neat proof for the Künneth formula of the local quotient of Nori's

fundamental group. This can be thought of as a new proof of [MS][Proposition 2.1] which is the key point for the proof of [MS][Theorem 2.3].

In the end of this paper, we give a counter example to show that [MS][Theorem 2.3] does not work if X or Y is not proper, where we take $X = \mathbb{A}_k^1$ and $Y = E$ to be a supersingular elliptic curve and k to be an algebraically closed field of characteristic 2. This also provides another counter example to show the failure of the exactness of the homotopy sequence for Nori's fundamental group (in the split case).

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2. THE GENERAL CRITERION

Definition 2.1. Let X be a reduced connected scheme over a field k , $x \in X(k)$ be a rational point. We call a triple $(P, G, p) \in N(X, x)$ a G -saturated torsor if the canonical map $\pi^N(X, x) \rightarrow G$ is surjective.

Remark. Here we are using the terminology in [EHV][Definition 3.2], where they defined a G -saturated bundle to be a pointed torsor $(P, G, p) \in N(X, x)$ with the property that $O_P(P) = k$. Nori has proved in [Nori][Part I, Chapter II, Proposition 3] that if X is (in addition) proper then the two definitions above are equivalent, where he called a G -saturated torsor "reduced" [Nori][Part I, Chapter II, Definition 3].

Definition. Let $f : X \rightarrow S$ be a map of schemes, \mathcal{F} be a sheaf of O_X -modules, $s : \text{Spec}(\kappa(s)) \hookrightarrow S$ a point, then we get a Cartesian diagram:

$$\begin{array}{ccc} X_s & \xrightarrow{t} & X \\ \downarrow g & & \downarrow f \\ \text{Spec}(\kappa(s)) & \xrightarrow{s} & S \end{array}$$

We say \mathcal{F} satisfies base change at s if the canonical map

$$s^* f_* \mathcal{F} \rightarrow g_* t^* \mathcal{F}$$

is surjective. Note that if f is proper, S is locally noetherian, \mathcal{F} is coherent and flat over S then \mathcal{F} satisfies base change at s if and only if the above canonical map is an isomorphism (see [Hart][Chapter III, Theorem 12.11]).

Theorem 2.2. (H.Esnault, P.H.Hai, E.Viehweg) *Let $f : X \rightarrow S$ be a separable proper morphism with geometrically connected fibres between two reduced connected locally noetherian schemes over a perfect field k . We suppose further that S is irreducible. Let $x \in X(k)$, $s \in S(k)$ and assume $f(x) = s$. Then the following conditions are equivalent:*

(1) the sequence

$$\pi^N(X_s, x) \rightarrow \pi^N(X, x) \rightarrow \pi^N(S, s) \rightarrow 1$$

is exact;

- (2) for any G -saturated torsor (P, G, p) with structure map $\pi : P \rightarrow X$, $\pi_* O_P$ satisfies base change at s and the image of the composition $\pi^N(X_s, x) \rightarrow \pi^N(X, x) \rightarrow G$ is a normal subgroup of G ;
- (3) for any G -saturated torsor (P, G, p) with structure map $\pi : P \rightarrow X$, $\pi_* O_P$ satisfies base change at s and there is a G' -saturated torsor $\pi' : P' \rightarrow S$ together with a morphism $(P, G) \xrightarrow{\theta} (P', G')$ satisfy that the θ -induced map $(\pi'_* O_{P'})_s \rightarrow (f_* \pi_* O_P)_s$ is an isomorphism.

Proof. "(1) \implies (2)" If the homotopy sequence is exact then clearly the image of $\pi^N(X_s, x) \rightarrow \pi^N(X, x) \rightarrow G$ (which is denoted by H) is normal in G . The exactness also gives us a commutative diagram

$$\begin{array}{ccc} \pi^N(X, x) & \longrightarrow & \pi^N(S, s) \\ \downarrow & & \downarrow \\ G & \longrightarrow & G/H \end{array}$$

This commutative diagram gives us a G/H -saturated torsor $(P', G/H, p')$ over S and a morphism in $N(X, x)$:

$$\lambda : (P, G, p) \rightarrow (P' \times_S X, G/H, p' \times_S X) \cong (P/H, G/H, p).$$

Let W' be the push forward of the structure sheaf of P' to S , $V := \pi_* O_P$, $W := f^* W'$. Let $\lambda^* : W \rightarrow V$ be the map induced by λ . If we pull-back λ^* to X_s then we get a morphism in the category of essentially finite vector bundles because $V|_{X_s}$ (resp. $W|_{X_s}$) is the 0-th direct image of the structure sheaf of the torsor $P \times_X X_s$ (resp. $P' \times_S X_s$). From [Nori][Part I, Chapter I, Proposition 2.9], this λ^* corresponds, via Tannakian duality, to the morphism

$$k[G]^{\pi^N(X_s, x)} = k[G]^H = k[G/H] \rightarrow k[G]$$

in the category of $\mathbf{Rep}_k(\pi^N(X_s, x))$. Hence $W|_{X_s}$ is the maximal trivial subbundle of $V|_{X_s}$. But $H^0(X_s, V|_{X_s}) \otimes_k O_{X_s} \subseteq V|_{X_s}$ is the maximal trivial sub embedding (see lemma 2.3 below), thus the canonical map

$$W|_{X_s} = H^0(X_s, W|_{X_s}) \otimes_k O_{X_s} \rightarrow H^0(X_s, V|_{X_s}) \otimes_k O_{X_s}$$

is an isomorphism. But note that the above map factors $W|_{X_s} \rightarrow f^* f_* V|_{X_s}$. This implies $f^* f_* V|_{X_s} \rightarrow H^0(X_s, V|_{X_s}) \otimes_k O_{X_s}$ is an isomorphism, so base change is satisfied.

"(2) \implies (3)" Let $H \subseteq G$ be the image of the composition $\pi^N(X_s, x) \rightarrow \pi^N(X, x) \rightarrow G$. Since it is normal we get a G/H -torsor P/H on X . If W is the push-forward of the structure sheaf of P/H to X and $V := \pi_* O_P$, then we know from our assumption that W and V satisfy base change at s . Let $\lambda : W \hookrightarrow V$ be the imbedding induced $P \rightarrow P/H$,

then we have the following commutative diagram of sheaves on X_s :

$$\begin{array}{ccccc} f^*f_*W|_{X_s} & \xrightarrow{a_1} & H^0(X_s, W|_{X_s}) \otimes_k O_{X_s} & \xrightarrow{a_2} & W|_{X_s} \\ \downarrow f^*f_*\lambda & & \downarrow H^0(X_s, \lambda|_{X_s}) & & \downarrow \lambda|_{X_s} \\ f^*f_*V|_{X_s} & \xrightarrow{a_3} & H^0(X_s, V|_{X_s}) \otimes_k O_{X_s} & \xrightarrow{a_4} & V|_{X_s} \end{array}$$

By base change a_1, a_3 are isomorphisms. Since $\lambda|_{X_s}$ corresponds via Tannakian duality to $k[G]^H \hookrightarrow k[G]$ (in the category $\mathbf{Rep}_k(\pi^N(X_s, x))$), $W|_{X_s}$ is imbedded as the maximal trivial subbundle of $V|_{X_s}$. Hence a_2 and $H^0(X_s, \lambda|_{X_s})$ are isomorphisms. So $f^*f_*\lambda$ is also an isomorphism. In particular

$$(f_*\lambda)_x : (f_*W)_x \rightarrow (f_*V)_x$$

is an isomorphism. Let $r \in \mathbb{N}$ be the rank of W . For any point $t \in S$, since

$$H^0(X_t, W|_{X_t}) \otimes_k O_{X_t} \rightarrow W|_{X_t}$$

is always an imbedding (lemma 2.3), we have $\dim_k(H^0(X_t, W|_{X_t})) \leq r$. But on the other hand, since W satisfies base change at s , $r = \dim_k(H^0(X_s, W|_{X_s}))$ reaches the minimal dimension (the dimension at the generic point), so by semi-continuity theorem we have

$$\dim_k(H^0(X_t, W|_{X_t})) \geq \dim_k(H^0(X_s, W|_{X_s})) = r.$$

This implies $H^0(X_t, W|_{X_t})$ has constant dimension r , and hence W satisfies base change all over S . So f_*W a vector bundle. Since $f^*f_*W \rightarrow W$ is injective after restricting to all the points of X , we have it is an embedding as a subbundle (i.e. injective and locally split). But since a_1, a_2 are isomorphisms, we have $f^*f_*W \rightarrow W$ is an isomorphism. Now we can check easily that $\mathrm{Spec}(f_*W) \rightarrow S$ with the canonical G/H -action induced from P/H is an FPQC-torsor which satisfies all our conditions in (3).

"(3) \implies (1)" Let N be the image of $\mathrm{Ker}(\pi^N(f))$ in G (where $\pi^N(f)$ is the map $\pi^N(X, x) \rightarrow \pi^N(S, s)$), N' be the kernel of $G \rightarrow G'$, and $H \subseteq G$ be the image of the composition $\pi^N(X_s, x) \rightarrow \pi^N(X, x) \rightarrow G$. We also write $W := \pi'_*O_{P'}$ and $V := \pi_*O_P$. We first note that the θ -induced map $f^*W|_{X_s} \rightarrow V|_{X_s}$ corresponds to $k[G/N'] \rightarrow k[G]$ in $\mathbf{Rep}_k(\pi^N(X_s, x))$. But from base change of V and the fact that the θ -induced map $W_x \rightarrow (f_*V)_x$ is an isomorphism we know that $f^*W|_{X_s} \rightarrow V|_{X_s}$ should be the same as $H^0(X_s, V|_{X_s}) \otimes_k O_{X_s} \rightarrow V|_{X_s}$ as subobjects. Thus the canonical imbedding $k[G/N'] \hookrightarrow k[G/H]$ should be an isomorphism. Hence $N' = H$ as subgroups. But since we have $H \subseteq N \subseteq N'$, so $H = N$ as well. Because the equality holds for all G -saturated torsor (P, G, p) , we have $\pi^N(X_s, x) \rightarrow \mathrm{Ker}(\pi^N(f))$ is surjective. This completes the proof. \square

Lemma 2.3. *If X is a reduced connected proper scheme over a perfect field k with a rational point $x \in X(k)$, then for any essentially finite vector bundle V on X the canonical morphism $\Gamma(X, V) \otimes_k O_X \rightarrow V$ imbeds $\Gamma(X, V) \otimes_k O_X$ as the maximal trivial subbundle of V .*

Proof. Let $\mathrm{Ess}(X)$ be the category of essentially finite vector bundles, $\omega_x : \mathrm{Ess}(X) \rightarrow \mathrm{Vec}_k$ be the fibre functor. Then applying ω_x to the canonical morphism $\Gamma(X, V) \otimes_k O_X \rightarrow$

V we get $\mathrm{Hom}_{O_X}(O_X, V) \cong \Gamma(X, V) \rightarrow V_x \otimes_{O_{X,x}} k = \omega_x(V)$. But note that we have $\mathrm{Hom}_{O_X}(O_X, V) \cong \mathrm{Hom}_{\pi^N(X,x)}(k, \omega_x(V))$ where k stands for the dim 1 vector space with trivial $\pi^N(X,x)$ action. One checks readily that under these isomorphisms we get exactly the canonical injection $\mathrm{Hom}_{\pi^N(X,x)}(k, \omega_x(V)) \rightarrow \omega_x(V)$ sending any morphism $k \rightarrow \omega_x(V)$ to the image of $1 \in k$. Since this map imbeds $\mathrm{Hom}_{\pi^N(X,x)}(k, \omega_x(V))$ as the maximal trivial sub of $\omega_x(V)$. Using Tannakian duality we get our result. \square

2.1. Application to the étale quotient.

Definition 2.4. Let X be a connected reduced locally noetherian scheme over a perfect field k which admits a rational point $x \in X(k)$. Let $N^{\mathrm{ét}}(X, x)$ be the full subcategory of $N(X, x)$ whose objects consist of those (P, G, p) with G finite étale. This sub category is filtered so we can define the étale quotient of $\pi^N(X, x)$ to be $\pi^{\mathrm{ét}}(X, x) := \varprojlim_{N^{\mathrm{ét}}(X, x)} G$.

We have an obvious surjection: $\pi^N(X, x) \twoheadrightarrow \pi^{\mathrm{ét}}(X, x)$.

Lemma 2.5. *Let X be a connected reduced scheme over a perfect field k which admits a rational point $x \in X(k)$. Let (P, G, p) be an étale torsor over (X, x) . This torsor is G -saturated if and only if P is connected.*

Proof. Since P has a rational point so connectedness is equivalent to geometrical connectedness, and also the formation of Nori's fundamental group is compatible with separable field extensions, thus we can reduce to the case when k is algebraically closed.

" \implies " Let's take $Q \subseteq P$ to be the connected component of P containing p . Now G is an abstract group we can write the action $\rho : P \times_k G \rightarrow P$ as $\coprod_G P \rightarrow P$ where each component in the direct union is mapped to P via a unique element in G . Since P is an G -torsor we have the following cartesian diagram:

$$\begin{array}{ccc} \coprod_G P & \xrightarrow{\rho} & P \\ \downarrow id^G & & \downarrow \\ P & \longrightarrow & X \end{array}$$

If we let $H \subseteq G$ be the maximal subgroup of G which fix Q , then we can see by definition that $Q \times_k H \subseteq P \times_k G$ is the intersection of $\rho^{-1}(Q)$ and $(id^G)^{-1}(Q)$. Thus the square

$$\begin{array}{ccc} \coprod_H Q & \xrightarrow{\rho} & Q \\ \downarrow id^H & & \downarrow \\ Q & \longrightarrow & X \end{array}$$

is cartesian. Hence Q is an H -torsor. But from the assumption the imbedding $H \rightarrow G$ should be surjective. This tells us $H = G$. But then the map of G -torsors $Q \subseteq P$ should also be an isomorphism. So P is connected.

" \impliedby " Let $(P', G', p') \rightarrow (P, G, p)$ be any morphism in $N(X, x)$. Since $P \rightarrow X$ is étale, we know $P' \rightarrow P$ is finite flat. Thus the image must be both open and closed, and hence it must be the whole of P . But if we pull-back the surjective map $P' \rightarrow P$ via $x \in X(k)$, we

will get the group homomorphism $G' \rightarrow G$. Thus this homomorphism must be surjective. Since (P', G', p') is taken arbitrarily, it actually shows that (P, G, p) is G -saturated. \square

Corollary 2.6. *Let $f : X \rightarrow S$ be a separable proper morphism with geometrically connected fibres between two reduced connected locally noetherian schemes over a perfect field k . Let $x \in X(k)$, $s \in S(k)$ and assume $f(x) = s$. Then the homotopy sequence:*

$$\pi^{\acute{e}t}(X_s, x) \rightarrow \pi^{\acute{e}t}(X, x) \rightarrow \pi^{\acute{e}t}(S, s) \rightarrow 1$$

is exact.

Proof. Without loss of generality one may assume $k = \bar{k}$ [Nori][Part I, Chapter II, Proposition 5]. Now let (P, G, p) be a G -saturated étale torsor over X , $\pi : P \rightarrow X$ be the structure map $V := \pi_* O_P$. Let $P \xrightarrow{\phi} Q \xrightarrow{\varpi} S$ be the stein factorization of the proper map $P \xrightarrow{\pi} X \xrightarrow{f} S$. Since $f \circ \pi$ is proper separable ϖ is finite étale. Thus ϕ is proper separable surjective with geometrically connected fibres. But then the pull back $\phi_s : P_s \rightarrow Q_s$ along the rational point $s \hookrightarrow S$ is also proper separable surjective with geometrically connected fibres. Hence $O_{Q_s} \rightarrow (\phi_s)_* O_{P_s}$ is an isomorphism. This tells us base change is satisfied for P at s .

The action $P \times_k G \rightarrow P$ induces a map $V \rightarrow V \otimes_k k[G]$. Push it to S we get $f_* V \rightarrow f_* V \otimes_k k[G]$. Thus there is an action of G on Q which makes ϕ G -equivariant. If we pull back the map $P \rightarrow Q \times_S X$ along the rational point $x \in X(k)$, we get a G -equivariant map $t : G \rightarrow G'$ (where the identity point e of G comes from p and G' is a G -set with a distinguished point q). One checks readily that $H := t^{-1}(q)$ is the stabilizer of $t(e)$, hence a subgroup of G . Now let $h \in H$ be an element. Considering the S -isomorphism $Q \rightarrow Q$ induced by h . Evidently h sends q to q , and since Q is a connected finite étale cover of S , the S -isomorphism induced by h must be the identity. Hence H acts trivially on Q and in particular it also acts trivially on G' . So for any $x \in G$, we have $t(e)hx^{-1} = t(x)hx^{-1} = t(x)x^{-1} = t(xx^{-1}) = t(e)$. As a consequence, H is a normal subgroup of G . But since $t : G \rightarrow G'$ is faithfully flat, we actually know that G' is the quotient of G by H (and t is the quotient map). Thus we get a commutative diagram:

$$\begin{array}{ccc} P \times_k G & \xrightarrow{\cong} & P \times_X P \\ \downarrow & & \downarrow \\ Q \times_k G' & \xrightarrow{\rho} & Q \times_S Q \end{array}$$

Let r be the degree of the connected finite étale cover $\varpi : Q \rightarrow S$. Then one sees easily that both $Q \times_k G'$ and $Q \times_S Q$ are finite étale of degree r . This shows that the Q -morphism ρ is finite étale of degree 1, and hence an isomorphism. Now $\varpi : Q \rightarrow S$ has a structure of a G' -torsor which satisfies all the conditions in (3) of our main theorem. So we can use the same argument we have used in "(3) \implies (1)" to conclude our theorem. \square

3. THE PROPER CASE

Theorem 3.1. (H.Esnault, P.H.Hai, E.Viehweg) *Let $f : X \rightarrow S$ be a proper separable morphism with geometrically connected fibres between two reduced connected proper schemes over a perfect field k , $x \in X(k)$, $s \in S(k)$, $f(x) = s$. Assume further that S is irreducible. Then the homotopy sequence*

$$\pi^N(X_s, x) \rightarrow \pi^N(X, x) \rightarrow \pi^N(S, s) \rightarrow 1$$

is exact if and only if for any G -saturated torsor $(P, G, p) \in N(X, x)$ with structure map $\pi : P \rightarrow X$, $V := \pi_ O_P$ satisfies base change at s and $f_* V$ is essentially finite.*

Proof. " \Leftarrow " Since $f_* V$ satisfies base change, the canonical map $f^* f_* V \rightarrow V$ is of the form

$$\Gamma(X_s, V|_{X_s}) \otimes_k O_{X_s} \rightarrow V|_{X_s}$$

after restricting to the fibre X_s . Because $f^* f_* V \rightarrow V$ is a map of essentially finite vector bundles, the kernel of it is also a vector bundle. But the kernel is trivial on X_s , so the kernel itself is trivial. Thus $f^* f_* V \subseteq V$ is a subobject in the category of essentially finite vector bundles on X and it becomes the maximal trivial subobject after restricting to X_s . Now let G' be the Tannakian group of the sub Tannakian category of $\text{Ess}(S)$ generated by $f_* V$. The imbedding $f^* f_* V \rightarrow V$ gives us a surjection $\lambda : G \rightarrow G'$. Let H be the kernel of λ . Then $f^* f_* V \rightarrow V$ corresponds via Tannakian duality to an inclusion $M \subseteq k[G]$ in $\text{Rep}_k(G)$. Note that since M comes from an object in $\text{Rep}_k(G')$ via $\lambda : G \rightarrow G'$, so $M \subseteq k[G]$ factors through the inclusion $k[G]^H \subseteq k[G]$. On the other hand, since we have a surjection $\pi^N(S, s) \rightarrow G'$, by [Nori][Chapter I, Proposition 3.11] we have a G' -saturated torsor $(P', G', p') \in N(S, s)$ with a map

$$\theta : (P, G, p) \rightarrow f^*(P', G', p')$$

in $N(X, x)$ extending λ . Let $V' := \pi'_* O_{P'}$, $\pi' : P' \rightarrow S$. Then since $P \rightarrow P' \cong P/H$ is faithfully flat, $f^* V' \subseteq V$ is a subbundle, and this subbundle corresponds via Tannakian duality to the inclusion $k[G]^H \subseteq k[G]$. But clearly $f^* V' \subseteq V$ factors $f^* f_* V \rightarrow V$, so $k[G]^H \subseteq k[G]$ factors $M \subseteq k[G]$, which means $k[G]^H = M$. So we have $V' \cong f_* V$. Now the triple (P', G', p') satisfies all our conditions in Theorem 2.2 (3), so we get the exact sequence.

" \Rightarrow " By Theorem 2.2 we have a G' -saturated torsor $(P', G', p') \in N(S, s)$ and a morphism

$$\theta : (P, G, p) \rightarrow f^*(P', G', p') \in N(X, x)$$

such that the induced map $V'_s \rightarrow (f_* V)_s$ is an isomorphism, where $V' := \pi'_* O_{P'}$ and $\pi' : P' \rightarrow S$ is the structure map. Because V satisfies base change at s , there is a neighborhood $s \in U$ such that $f_* V$ is a vector bundle on U and the adjunction map $f^* f_* V \rightarrow V$ is a sub bundle (locally split) on $f^{-1}(U)$. But $f^* V' \rightarrow V$ is a sub vector bundle and $f^* V' \rightarrow V$ factors through the adjunction map, so $f^* V' \rightarrow f^* f_* V$ is a sub vector bundle on $f^{-1}(U)$. Since $V'_s \cong (f_* V)_s$, $f^* V' \rightarrow f^* f_* V$ is an isomorphism on $f^{-1}(U)$.

Hence the injective map $V' \rightarrow f_*V$ is also an isomorphism on U . Now by [EGA][Théorème 7.7.6] there is a coherent sheaf \mathcal{Q} on S such that

$$f_*(V/f^*V') \cong \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{O}_S).$$

Since locally $\mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{O}_S)$ is contained in a vector bundle and we have

$$f_*V/V' \subseteq f_*(V/f^*V') = \mathcal{H}om_{\mathcal{O}_S}(\mathcal{Q}, \mathcal{O}_S),$$

so if there is $t \in S \setminus U$ such that $(f_*V/V')_t \neq 0$, then we can choose an open affine $t \in \text{Spec}(A) \subseteq S$ such that

$$(f_*V/V')|_{\text{Spec}(A)} \subseteq \bigoplus_{i=0}^n A_i,$$

where A_i is a rank 1 free A -module for all $0 \leq i \leq n$. Notice that since S is integral $\text{Spec}(A)$ is non-empty, so A is an integral ring. This implies f_*V/V' is non-zero at the generic point which contradicts to the fact that f_*V/V' has support in $S \setminus U$. So $V' \rightarrow f_*V$ is an isomorphism on S . But V' is certainly essentially finite. This completes the proof. \square

3.1. Application to the Künneth formula.

Definition 3.2. Let X be a reduced connected scheme over a field k with a rational point $x \in X(k)$. Let $N^F(X, x)$ be the full subcategory of $N(X, x)$ whose objects consist of pointed torsors with finite local groups. This category is also filtered so we can write $\pi^F(X, x) := \varprojlim_{N^F(X, x)} G$. If X is also proper and k is perfect, then $\pi^F(X, x)$ is the Tannakian group of the full subcategory of the category of essentially finite vector bundles $\text{Ess}(X)$ consisting of F -trivial bundles, i.e. vector bundles which are trivial after pull back along some relative Frobenius $\phi_{(-t)} : X^{(-t)} \rightarrow X$ with $t \in \mathbb{N}$.

Corollary 3.3. *Let X and Y be two reduced connected proper schemes over a perfect field k . Let $x \in X(k)$, $y \in Y(k)$. Then the canonical map*

$$\pi^F(X \times_k Y, (x, y)) \rightarrow \pi^F(X, x) \times_k \pi^F(Y, y)$$

is an isomorphism of k -group schemes.

Proof. We will use the obvious analogues of Theorem 3.1 to prove this theorem. Note that after replacing $\pi^N(X, x)$ by $\pi^F(X, x)$, "torsor" by "local torsor" (torsors whose groups are local), essentially finite vector bundle by F -trivial vector bundle, Theorem 2.2 and Theorem 3.1 are still true.

To prove this corollary we only need to show that the sequence

$$1 \rightarrow \pi^F(Y, y) \rightarrow \pi^F(X \times_k Y, (x, y)) \rightarrow \pi^F(X, x) \rightarrow 1$$

is exact. So we have to check that for any G -saturated local torsor $(P, G, p) \in N^F(X, x)$, $V := \pi_*\mathcal{O}_P$ ($\pi : P \rightarrow X$ is the structure map) satisfies base change at x and f_*V is an F -trivial vector bundle.

Now suppose that V is trivialized by

$$X^{(-t)} \times_k Y^{(-t)} = (X \times_k Y)^{(-t)} \rightarrow X \times_k Y.$$

Consider the following commutative diagram

$$\begin{array}{ccccc} Y & \xrightarrow{y} & X^{(-t)} \times_k Y & \xrightarrow{p_1} & X^{(-t)} \\ \parallel & & \downarrow \phi_{(-t)} \times id & & \downarrow \phi_{(-t)} \\ Y & \xrightarrow{y} & X \times_k Y & \xrightarrow{f} & X \end{array}.$$

Let W be the pull back of V via $X^{(-t)} \times_k Y \rightarrow X \times_k Y$. Since W has trivial fibres along the projection $p_2 : X^{(-t)} \times_k Y \rightarrow Y$ and $X^{(-t)}$ is proper separable and geometrically connected scheme, so there exists a vector bundle on E on Y such that $p_2^*E \cong W$, so V has constant fibres along $f : X \times_k Y \rightarrow X$. Consequently base change is satisfied for V along f (at any point of X). On the other hand we have the following trivial cartesian diagram

$$\begin{array}{ccc} X \times_k Y & \xrightarrow{p_2} & Y \\ \downarrow p_1 & & \downarrow b \\ X & \xrightarrow{a} & \text{Spec}(k) \end{array},$$

where a and b are structure maps. Because of base change we have $a^*b_*E \cong p_{1*}p_2^*E$. This implies $p_{1*}W = p_{1*}p_2^*E$ is a trivial vector bundle. But since $\phi_{(-t)} : X^{(-t)} \rightarrow X$ is faithfully flat, so we have a canonical isomorphism

$$p_{1*}W = p_{1*}(\phi_{(-t)} \times id)^*V \cong \phi_{(-t)}^*f_*V.$$

Thus $\phi_{(-t)}^*f_*V$ is a trivial vector bundle. By definition f_*V is F -trivial. \square

Remarks 3.4. (1) Here we didn't assume X or Y is irreducible, this is because we have only used the sufficiency part of Theorem 3.1 in which only the citation of Theorem 2.2 used the irreducibility. But we only used (3) \Rightarrow (1) part of Theorem 2.2 where the irreducibility plays no role.

(2) This corollary gives another way to see [MS][Proposition 2.1] which is the key point in the proof of the Künneth formula for Nori's fundamental group. But unfortunately, for the full proof of of Künneth formula we have to use the same trick employed in [MS] to reduce the problem for π^N to the problem for π^F . At the moment, I can not find any easy way to reduce the problem to π^F using our language here.

4. THE SMOOTH CASE

Definition 4.1. Let X be a smooth geometrically connected scheme over a perfect field k of positive characteristic. For $i \in \mathbb{N}$, we have the relative the relative Frobenius $\phi_i : X^{(i)} \rightarrow X^{(i+1)}$ starting with $X^{(0)} = X$. Let $t \in \mathbb{N}$. A t -stratified bundle $(E_i, \sigma_i, i \in \mathbb{N})$ consists of a sequence of vector bundles E_i on $X^{(i)}$, and a sequence of $O_{X^{(i)}}$ -isomorphisms

$$\sigma_i : E_i \rightarrow \phi_i^*E_{i+1}$$

for all $i \geq 1$ and for $i = 0$,

$$\sigma_0 : \phi_{(-t,0)}^*E_i \rightarrow \phi_{(-t,1)}^*E_{i+1}$$

is an $O_{X(-t)}$ -isomorphism, where $\phi_{(-t,0)} : X^{(-t)} \rightarrow X^{(0)}$ is the composition of relative Frobenius and similarly for $\phi_{(-t,1)}$.

Definition 4.2. Let X be a smooth geometrically connected scheme over a perfect field k of positive characteristic, $t \in \mathbb{N}$. Let $\text{Strat}(X, t)$ be the category whose objects consist of t -stratified bundles, whose morphisms

$$\text{Hom}((E_i, \sigma_i, i \in \mathbb{N}), (F_i, \tau_i, i \in \mathbb{N}))$$

are the set of morphisms $E_i \rightarrow F_i$ for $i \in \mathbb{N}$ which are compatible with σ_i and τ_i in a natural way. Because of the faithful flatness of the relative Frobenius, one has a fully faithful imbedding $\text{Strat}(X, t) \subseteq \text{Strat}(X, t+1)$. Now taking the 2-direct limit in the category of categories, one gets a new category $\text{Strat}(X, \infty) := \varinjlim_{t \geq 0} \text{Strat}(X, t)$.

Theorem 4.3. (Esnault and Hogadi). *Let X be a smooth connected scheme over a perfect field k of positive characteristic with a rational point $x \in X$, $t \in \mathbb{N}$. Then the categories $\text{Strat}(X, t)$, $\text{Strat}(X, \infty)$ together with the fibre functor $(E_i, \sigma_i, i \in \mathbb{N}) \mapsto E_0|_x$ are neutral Tannakian categories. If we set $\text{Strat}^{\text{fin}}(X)$ to be the full subcategory of $\text{Strat}(X, \infty)$ whose objects consists of those whose Tannakian groups are finite, then the Tannakian group of $\text{Strat}^{\text{fin}}(X)$ is isomorphic to $\pi^N(X, x)$ as k -group schemes.*

Theorem 4.4. *If S is a connected scheme smooth over a perfect field k of positive characteristic, $f : X \rightarrow S$ is a smooth proper morphism with geometrically connected fibres, $s \in S(k)$, $x \in X(k)$ such that $f(x) = s$, then the homotopy sequence*

$$\pi^N(X_s, x) \rightarrow \pi^N(X, x) \rightarrow \pi^N(S, s) \rightarrow 1$$

is exact if and only if for any G -saturated torsor $(P, G, p) \in N(X, x)$ with structure map $\pi : P \rightarrow X$, $V_0 := \pi_ O_P$ satisfies base change at s and there is a neighborhood U of s and an object $(W_i, \tau_i, i \geq 0) \in \text{Strat}^{\text{fin}}(U/k)$ which satisfies*

- (1) $W_0 = f_* V_0|_U$;
- (2) *there is an imbedding $f^*(W_i, \tau_i, i \geq 0) \subseteq (V_i, \sigma_i, i \geq 0)|_{f^{-1}(U)}$ such that $f^*W_0 \rightarrow V_0|_{f^{-1}(U)}$ is the canonical map $f^*f_*\pi_* O_P \rightarrow \pi_* O_P$ restricting to $f^{-1}(U)$, where $(V_i, \sigma_i, i \geq 0) \in \text{Strat}^{\text{fin}}(X)$ is the stratified object corresponding to (P, G) ([EH][§4, Construction 4.1]).*

Proof. " \implies " According to [EPS][Appendix A.1 (iii) (a)(b)], there is an object $(W_i, \sigma_i, i \geq 0) \in \text{Strat}^{\text{fin}}(S/k)$ which satisfies

- (1) there is an imbedding $f^*(W_i, \tau_i, i \geq 0) \subseteq (V_i, \sigma_i, i \geq 0)$;
- (2) if we restrict the imbedding $f^*W_0 \rightarrow V_0$ to X_s then it gives the maximal trivial subbundle of $V_0|_{X_s}$.

In other words $f^*W_0|_{X_s}$ is equal to $H^0(X_s, V_0|_{X_s}) \otimes_k O_{X_s}$ as subbundles of $V_0|_{X_s}$ (see Lemma 2.3). Thus the composition of maps

$$f^*W_0|_{X_s} \rightarrow f^*f_*V_0|_{X_s} \rightarrow H^0(X_s, V_0|_{X_s}) \otimes_k O_{X_s}$$

is an isomorphism. So each arrow is an isomorphism. Thus f_*V_0 has base change at s , so it is a vector bundle in a neighborhood U of s , and the map $W_0 \rightarrow f_*V_0$ is an isomorphism on U . Hence the stratified sheaf $(W_i, \sigma_i, i \geq 0)|_{U/k}$ satisfies our conditions.

" \Leftarrow " Let G' be the Tannakian group of the Tannakian category generated by $(W_i, \tau_i, i \geq 0) \in \text{Strat}^{\text{fin}}(U)$ with fibre functor s^* . Then we get a surjection $G \rightarrow G'$ because of the embedding condition (2). Moreover one has the following commutative diagram

$$\begin{array}{ccccccc} \pi^N(X_s, x) & \longrightarrow & \pi^N(f^{-1}(U), x) & \longrightarrow & \pi^N(U, s) & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ H & \longrightarrow & G & \longrightarrow & G' & \longrightarrow & 1 \end{array}$$

where H denotes the image of $\pi^N(X_s, x)$ in G . Now let V be an object in $\text{Rep}_k(G')$ corresponding to $(W_i, \tau_i, i \geq 0)$. Then by our condition (2) we have an imbedding $V \rightarrow k[G]$ in $\text{Rep}_k(G)$. Again by condition (2) together with base change we have $V = k[G]^H$ as subobjects of $k[G]$ in $\text{Rep}_k(\pi^N(X_s, x))$ (see also Lemma 2.3). But if we denote the kernel of $G \rightarrow G'$ by N then we have $V \subseteq k[G]^N$ (since V was in $\text{Rep}_k(G')$). Thus the canonical inclusion $k[G]^N \subseteq k[G]^H$ is an isomorphism. This shows $H = N$. So the image H is a normal subgroup of G . This together with base change at s implies the exactness (Theorem 2.2). \square

4.1. A counter example. Now consider k an algebraically closed field of characteristic 2, $X = \mathbb{A}_k^1$, $Y = E$ a supersingular elliptic curve. In the following we will construct an object

$$W := (W_i, \sigma_i, i \geq 0) \in \text{Strat}^{\text{fin}}(X \times_k Y)$$

(this object is in fact induced from an α_2 -torsor, see Remark 4.5 below) such that W is trivial after restricting to the fibres of the the projection $p_2 : X \times_k Y \rightarrow Y$. If K nneth formula was true for some point $x \in X(k)$ and $y \in Y(k)$, then one can see directly by using Tannakian duality (or if you like [EPS][Appendix A.1 (iii) (a)(b)]) that there exists

$$W' := (W_i, \sigma_i, i \geq 0) \in \text{Strat}^{\text{fin}}(Y)$$

and an imbedding $p_2^*W' \subseteq W$ such that the imbedding gives the maximal trivial sub object after restricting to the fibre of y under p_2 . Since the W is trivial on the fibre of y , the imbedding $p_2^*W' \subseteq W$ must be an isomorphism. Hence along the projection $p_1 : X \times_k Y \rightarrow X$ the fibres of W should all be isomorphic to W' . But we will see that the object W which we have constructed is not fibrewise constant.

Construction. Suppose $\pi : P \rightarrow Y$ is a non-trivial α_2 -tosor over Y . Let $V := \pi_*O_P$. Let \mathcal{L} be the cokernel of the structure map $O_Y \rightarrow V$, then \mathcal{L} is an essentially finite line bundle, so it has degree 0. If \mathcal{L} was not O_Y , then $H^1(Y, \mathcal{L}^{-1}) = \text{Ext}^1(O_Y, \mathcal{L}^{-1}) \neq 0$, so by Riemann-Roch $h^0(Y, \mathcal{L}^{-1}) = h^1(Y, \mathcal{L}^{-1}) \neq 0$. But this implies \mathcal{L}^{-1} is O_Y which is impossible. Hence we have $\mathcal{L} \cong O_Y$. This gives us an exact sequence

$$0 \rightarrow O_Y \rightarrow V \rightarrow O_Y \rightarrow 0.$$

We know that for $i = -1$, $P \rightarrow Y$ has already become a trivial torsor after pulling back along the relative Frobenius $\phi_{(-1)} : Y^{(-1)} \rightarrow Y$. Thus after choosing a section $Y^{(-1)} \rightarrow P \times_Y Y^{(-1)}$ we get an $Y^{(-1)}$ -scheme isomorphism $P \times_k Y^{(-1)} \cong \alpha_2 \times_k Y^{(-1)}$ which gives us a trivialization $\delta : \phi_{(-1)}^* V \cong O_{Y^{(-1)}} \oplus O_{Y^{(-1)}}$ making the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & O_{Y^{(-1)}} & \longrightarrow & \phi_{(-1)}^* V & \longrightarrow & O_{Y^{(-1)}} \longrightarrow 0 \\ & & \parallel & & \cong \downarrow \delta & & \parallel \\ 0 & \longrightarrow & O_{Y^{(-1)}} & \longrightarrow & O_{Y^{(-1)}} \oplus O_{Y^{(-1)}} & \longrightarrow & O_{Y^{(-1)}} \longrightarrow 0 \end{array}$$

commutative. By Grothendieck's FPQC descent theory there is an essentially unique descent isomorphism ε corresponding to V

$$\begin{array}{ccc} p_1^* \phi_{(-1)}^* V & \xrightarrow{\cong} & p_2^* \phi_{(-1)}^* V \\ \downarrow p_1^* \delta & & \downarrow p_2^* \delta \\ O_{Y^{(-1)} \times_Y Y^{(-1)}} \oplus O_{Y^{(-1)} \times_Y Y^{(-1)}} & \xrightarrow{\varepsilon} & O_{Y^{(-1)} \times_Y Y^{(-1)}} \oplus O_{Y^{(-1)} \times_Y Y^{(-1)}} \end{array},$$

where p_1 and p_2 are the two projections of $Y^{(-1)} \times_Y Y^{(-1)}$. This ε is expressible by a matrix in $GL_2(\Gamma(Y^{(-1)} \times_Y Y^{(-1)}, O_{Y^{(-1)} \times_Y Y^{(-1)}}))$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

Since P is not a trivial torsor $a \neq 0$. Moreover ε can also be seen as an $O_{Y^{(-1)} \times_Y Y^{(-1)}}$ -automorphism of $O_{Y^{(-1)} \times_Y Y^{(-1)}}[T]/T^2$ (where T is an indeterminate). An easy computation using the above matrix shows that $a^2 = 0$.

Let x be the indeterminate in $X = \mathbb{A}_k^1 = \text{Spec}(k[x])$. Then the 2×2 -matrix:

$$\begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}$$

in $GL_2(\Gamma(X \times_k Y^{(-1)} \times_Y Y^{(-1)}, O_{X \times_k Y^{(-1)} \times_Y Y^{(-1)}}))$ determines an isomorphism

$$\varepsilon' : O_{X \times_k Y^{(-1)} \times_Y Y^{(-1)}} \oplus O_{X \times_k Y^{(-1)} \times_Y Y^{(-1)}} \cong O_{X \times_k Y^{(-1)} \times_Y Y^{(-1)}} \oplus O_{X \times_k Y^{(-1)} \times_Y Y^{(-1)}}.$$

One checks readily that the pair

$$(O_{X \times_k Y^{(-1)}} \oplus O_{X \times_k Y^{(-1)}}, \varepsilon')$$

gives us a descent data (i.e. the cocycle condition is satisfied). This descent data will give us a rank 2 vector bundle W on $X \times_k Y$ with a trivialization $\xi : (id \times \phi_{(-1)})^* W \cong O_{X \times_k Y^{(-1)}} \oplus O_{X \times_k Y^{(-1)}}$ on $X \times_k Y^{(-1)}$. Let λ be the pull back of ξ along the relative Frobenius $X^{(-1)} \times_k Y^{(-1)} \rightarrow X \times_k Y^{(-1)}$. Then the pair (W, λ) is a 1-stratified bundle over $X \times_k Y$:

We set $W_0 := W$, $W_i = O_{X^{(i)}} \oplus O_{X^{(i)}}$ for $i \geq 1$ and

$$\sigma_0 = \lambda : \phi_{(-1)}^* W \cong O_{X^{(-1)} \times_k Y^{(-1)}} \oplus O_{X^{(-1)} \times_k Y^{(-1)}},$$

$\sigma_i = id : O_{X^{(i)}} \rightarrow O_{X^{(i)}}$. Now $(W_i, \sigma_i, i \geq 0)$ is a 1-stratified bundle which will be denoted by (W, λ) .

It is clear that this W does not have constant fibre along the projection $X \times_k Y \rightarrow X$ (the fibre along $x = 0$ splits while the fibre along $x = 1$ does not). We still have to check $(W, \lambda) \in \text{Strat}^{\text{fin}}(X \times_k Y)$, i.e. the Tannakian group which corresponds to (W, λ) is a finite group scheme.

Now we consider the tensor product $(W, \lambda) \otimes (W, \lambda)$ and the direct sum $(W, \lambda) \oplus (W, \lambda)$. It is enough to show that $(W, \lambda) \otimes (W, \lambda) \cong (W, \lambda) \oplus (W, \lambda)$, and hence (W, λ) sits in our category $\text{Strat}^{\text{fin}}(X \times_k Y)$. We see that $W \otimes_{O_{X \times_k Y}} W$ corresponds to the descent data

$$A := \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax & ax & a^2x^2 \\ 0 & 1 & 0 & ax \\ 0 & 0 & 1 & ax \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax & ax & 0 \\ 0 & 1 & 0 & ax \\ 0 & 0 & 1 & ax \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

On the other hand $W \oplus W$ corresponds to the following descent data:

$$B := \begin{pmatrix} 1 & ax & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & ax \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now let

$$C := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we get an automorphism of the trivial bundle $O_{X \times_k Y}^{\oplus 4}(-1)$. Since we have $CA = BC$, it follows that C induces an isomorphism between the descent data of $W \otimes_{O_{X \times_k Y}} W$ and $W \oplus W$. So we get an isomorphism $\theta : W \otimes_{O_{X \times_k Y}} W \rightarrow W \oplus W$. It is immediate from the definition of $\lambda \otimes \lambda$ and $\lambda \oplus \lambda$ that θ is indeed an isomorphism:

$$\theta : (W, \lambda) \otimes (W, \lambda) \cong (W, \lambda) \oplus (W, \lambda).$$

Remark 4.5. The object (W, λ) we have constructed is actually from an α_2 -torsor over $X \times_k Y$, i.e. there is an α_2 -torsor Q over $X \times_k Y$ such that the corresponding stratified sheaf (see [EH][Construction 4.1]) is (W, λ) . In fact the descent data

$$\begin{pmatrix} 1 & ax \\ 0 & 1 \end{pmatrix}$$

is not only a descent data of $O_{X \times_k Y(-1) \times_Y Y(-1)}$ -modules but also a descent data of algebras, or equivalently a descent data for affine schemes. Thus W is a coherent sheaf of $O_{X \times_k Y}$ -algebra. Let $Q = \text{Spec}(W)$, then the matrix A is the descent data for the $X \times_k Y$ -scheme $Q \times_{X \times_k Y} Q$ and the matrix B is the descent data for the $X \times_k Y$ -scheme $Q \times_k \alpha_2$. One can check easily that the map θ we constructed above is a morphism of descent data for affine schemes. Then we get an $X \times_k Y$ -isomorphism: $\rho : Q \times_k \alpha_2 \rightarrow Q \times_{X \times_k Y} Q$. One

can check that the composition of ρ with the second projection of $Q \times_{X \times_k Y} Q$ defines an action of α_2 and the composition of ρ with the first projection of $Q \times_{X \times_k Y} Q$ is the first projection of $Q \times_k \alpha_2$, because they are the case after pulling back to $X \times_k Y^{(-1)}$. This defines for us the α_2 -torsor Q .

Remark 4.6. Hélène Esnault and Andre Chatzistamatiou pointed to us the following improvement of the above example. Thanks to their suggestion our counter example may work for any characteristic $p > 0$. Now we consider the exact sequence of abelian sheaves in the flat topology

$$0 \rightarrow \alpha_p \rightarrow \mathbb{G}_a \xrightarrow{F} \mathbb{G}_a \rightarrow 0,$$

we then get a long exact sequence of abelian groups:

$$\cdots \rightarrow H_{fl}^0(-, \mathbb{G}_a) \rightarrow H_{fl}^1(-, \alpha_p) \rightarrow H_{fl}^1(-, \mathbb{G}_a) \xrightarrow{H^1(F)} H_{fl}^1(-, \mathbb{G}_a) \rightarrow \cdots$$

If we put $-$ to be an elliptic curve E and if there map $H^1(F)$ has none trivial kernel (like in our case), then we can choose some $a \neq 0$ in the that kernel. If we put $-$ to be $\mathbb{A}_k^1 \times_k E$ then

$$a \otimes x \in H_{fl}^1(\mathbb{A}_k^1 \times_k E, \mathbb{G}_a) = H_{fl}^1(E, \mathbb{G}_a) \otimes_k k[x].$$

If we choose an element $b \in H_{fl}^1(\mathbb{A}_k^1 \times_k E, \alpha_p)$ such that $b \mapsto a \otimes x$, then b is an α_p -torsor with non-constant fibres along the projection $\mathbb{A}_k^1 \times_k E \rightarrow \mathbb{A}_k^1$ because b has trivial image at $x = 0$ and non-trivial image at $x = 1$. This can not happen if Künneth formula was true.

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